Rapidity Divergences and Endpoint Region Deep Inelastic Scattering

Ou Zhang
University of Arizona

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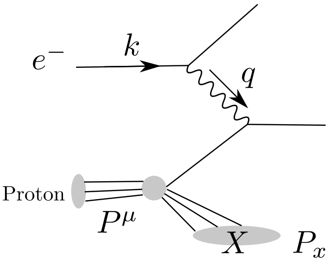
Deep Inelastic Scattering Kinematics



$$x = \frac{-q^2}{2p\cdot q} = \frac{Q^2}{2p\cdot q}$$

Light cone coordinates:

$$P^\mu = \tfrac{1}{2}\bar{n}\cdot P n^\mu + \tfrac{1}{2}n\cdot P\bar{n}^\mu + P_\perp^\mu$$
$$= \tfrac{1}{2}P^- n^\mu + \tfrac{1}{2}P^+\bar{n}^\mu + P_\perp^\mu$$

Target rest frame:

$\vec{q}_\perp = 0 \quad Q^2 = -q^+q^-$ taking $q^+ \gg q^-$

$$P = (M_p, M_p, 0) \quad x = -\frac{q^+q^-}{p^+q^- + p^-q^+} = -\frac{q^-}{p^-}$$

$$q = \left(\frac{Q^2}{xp^-}, -xp^-, 0\right)$$

$$p_x = \left(\frac{Q^2}{xp^-}, (1-x)p^-, 0\right)$$

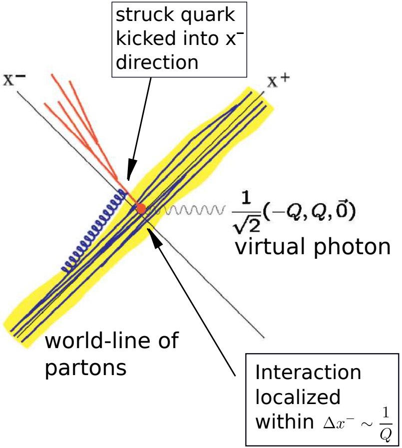
Boosting along $\hat{z}$ Breit Frame: $q = (-Q, Q, 0)$

$$P = \left(-\frac{M_p^2}{Q}x, -\frac{Q}{x}, 0\right)$$

$$P_x = \left(-Q, Q\left(1-\frac{1}{x}\right), 0\right)$$

DIS, Parton Model and Lightcone Coordinates

- Lightcone Coordinates fast-moving electron, proton, $m_p \sim m_e \sim 0, (p_i^+, p_i^-, \vec{0})$



struck quark kicked into x⁻ direction

$\frac{1}{\sqrt{2}}(-Q, Q, \vec{0})$ virtual photon

world-line of partons

Interaction localized within $\Delta x^- \sim \frac{1}{Q}$

- rest frame $\Delta x^+ \sim \Delta x^- \sim \frac{1}{m_p}$

Boost

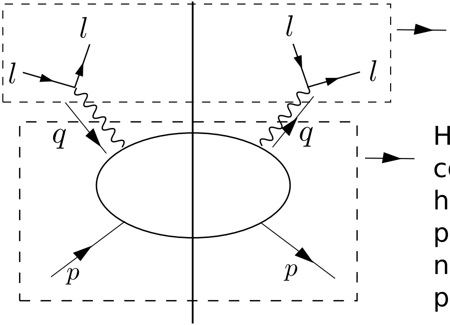
- Breit frame

  $\Delta x^+ \sim \frac{1}{m}\frac{Q}{m} = \frac{Q}{m^2}$ Large

  $\Delta x^- \sim \frac{1}{m}\frac{m}{Q} = \frac{1}{Q}$ Small

- Interactions between partons are spread out inside a fast moving proton

QCD Factorization for Deep Inelastic Scattering



Lepton tensor

$$L^{\mu\nu} = 2(k^\mu k'^\nu + k^\nu k'^\mu - g^{\mu\nu} k \cdot k')$$

Hadronic tensor $W^{\mu\nu}$:
contains information about
hadronic structure
perturbation QCD and
non-perturbative
parton distribution

- First analysis of DIS does not require any knowledge about QCD

$$d\sigma = \underbrace{\frac{d^3\vec{k'}}{2|\vec{k'}|}}_{\text{phase space scat. leptons}} \underbrace{\frac{4\pi\alpha^2}{s}}_{\text{EW vertices}} \underbrace{\frac{1}{Q^4}}_{\text{photon propagator}^2} L^{\mu\nu}(k,q) W_{\mu\nu}(p,q)$$

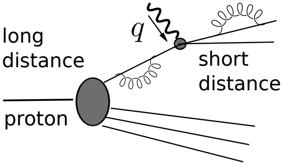
QCD Factorization for DIS towards parton model

- Hadroinc Tensor: symmetries (parity, Lorentz), hermiticity $W^{\mu\nu}=W^{\nu\mu\,*}$, current conservation $q_\mu W^{\mu\nu}=0$

$$W_{\mu\nu}=-\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)F_1(x,Q^2)+\left(p_\mu-q_\mu\frac{p\cdot q}{q^2}\right)\left(p_\nu-q_\nu\frac{p\cdot q}{q^2}\right)\frac{1}{p\cdot q}F_2(x,Q^2)$$

structure function

- Factorization: separating short distance (perturbative) process from long distance (non-perturbative) process by a chosen scale



$$\begin{aligned}F_2(x,Q^2)&=2xF_1(x,Q^2)\\&=x\sum_{i=q,\bar{q}}e_i^2\int_x^1\frac{d\xi}{\xi}\overbrace{f_i(\xi,\mu_f^2)}^{\text{parton distribution function}}\overbrace{H^i\left(\frac{x}{\xi},Q^2,\mu_f,\alpha_s(\mu_r)\right)}^{\text{computable QCD coefficient}}\end{aligned}$$

QCD Factorization: Perturbative QCD

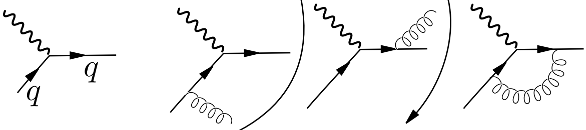
- The general structure of the calculatable coefficient the corrections looks like:

renormalization scale

finite pieces independent of Q^2

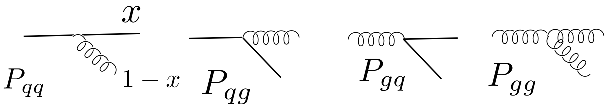
$$H^i_{(q)}\left(\frac{x}{\xi}, Q^2, \mu_f, \mu_r, \alpha_s\right) = \delta\left(1-\frac{x}{\xi}\right) + \frac{\alpha_s(\mu_r)}{4\pi}\left[P_{qq}\left(\frac{x}{\xi}\right)\ln\frac{Q^2}{mq^2} + C_2^q(x)\right]$$

quark parts



P_{ij}: Splitting Function
probability that a parton j splits collinearly into a parton i carrying a momentum fraction x

Large logarithms from collinear emissions



QCD Factorization: Parton Distribution Function to All α_s Orders

- The physical structure function is independent of μ_f

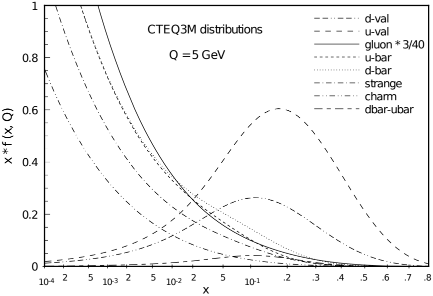
$$F_2(x,Q^2) = x\sum_{i=q,\bar{q}} e_i^2 \int_x^1 \frac{d\xi}{\xi} f_i(\xi,\mu_f^2)\left[\delta\left(1-\frac{x}{\xi}\right) + \underbrace{\frac{\alpha_s(\mu_r)}{2\pi}\left[P_{qq}\left(\frac{x}{\xi}\right)\ln\frac{Q^2}{\mu_f^2} + (C_2^q - Z_{qq})\left(\frac{x}{\xi}\right)\right]}\right]$$

Both pdf's and the short distance coefficients depend on μ_f

Short-distance Wilson coefficient

- The choice of μ_f : shifting terms between long- and short- distance parts
  Redefining non-perturbative and perturbative physics!

Parton Distribution Function to All α_s Orders



- In full glory (including gluons) the DGLAP eqs. read

$$\frac{d}{d\ln\mu}\begin{pmatrix} f_q(x,\mu) \\ f_g(x,\mu) \end{pmatrix} = \int_x^1 \frac{dz}{z}\begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix}_{(Z,\alpha_s)} \cdot \begin{pmatrix} f_q(x/Z,\mu) \\ f_g(x/Z,\mu) \end{pmatrix}$$

calculable in pQCD

- In real world, we measure cross-section (F_2) in experiment, calculate H^i in perturbative QCD, running RGE to sum up all α_s in f_i and extract a precise parton distribution function and its evolution

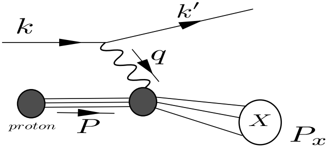
End-point Region $x \to 1$ in Parton Distribution Function

Review the scales in D.I.S. process

$$P_x^{\mu} = P^{\mu} + q = \frac{Q^2}{x}(1-x) + m_p^2$$

P_x^2	$\left(\frac{1}{x}-1\right)$	
$\sim Q^2$	~ 1	$x \ll 1$
$\sim \Lambda_{\rm QCD}^2$	$\sim \Lambda^2/Q^2 \sim 0$	$x = 1$
$\sim Q\Lambda$	$\sim \Lambda/Q$	$x \sim 1$

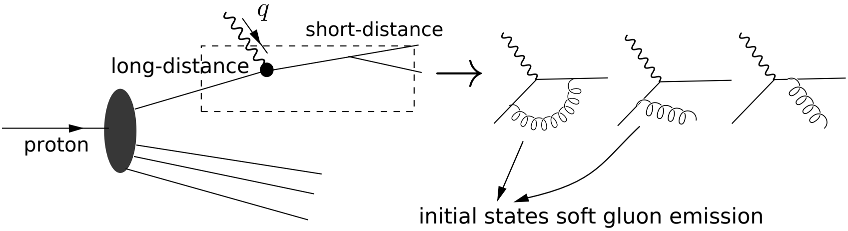
- inclusive process: H^i can be computed as operator product expansion in pQCD
- exclusive process:
  $\Lambda_{\rm QCD}$ is strong enough to confine all partons together to hadrons
  $e^- p \to e^- p' \leftarrow$ eg. excited proton resonance region
- end-point region semi-inclusive process
  Q is not large enough to struck parton out of proton, $\Lambda_{\rm QCD}$ the interaction between partons can not be ignored



x~0.8 10 GeV
x~0.5 for LHC

Analyzing Non-perturbative Effects at Endpoint Region $x \to 1$

- When $\Lambda_{\rm QCD}$ is taken into account, the Infrared Divergence caused by the soft gluon emission from initial state partons can not be cancelled entirely

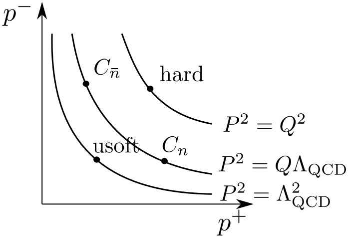


with the form $H^i \sim \int_0^1 \frac{dx}{x}\left(\frac{1}{1-x}\right)$ with Endpoint Singularity

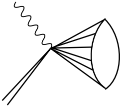
- Calling for effective field theories to describe non-perturbative effects -- SCET

Soft Collinear Effective Theory I and II

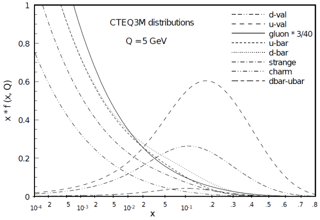
$\mathrm{SCET_I}$



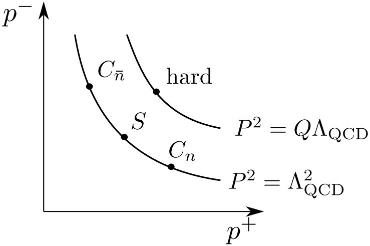
DIS small x inclusive



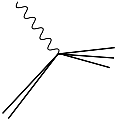
energetic jets



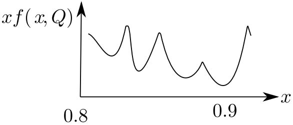
$\mathrm{SCET_{II}}$



DIS $x \to 1$ semi-inclusive



Narrow final states



Deep Inelastic Scattering Factorization in SCET

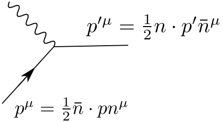
- DIS Hadronic Tensor

QCD	Q^2	$W = f_{\text{P.D.F.}} \otimes H_{\text{QCD perturbation}}$
$\downarrow$ SCET$_I$	$Q\Lambda_{\text{QCD}}$	$W = f_{\text{P.D.F.}} \otimes J(Q\Lambda) \otimes S(\Lambda^2_{\text{QCD}}) \otimes H_{\text{QCD coefficients}}(Q^2)$
$\downarrow$ SCET$_{II}$	Λ^2_{QCD}	$W = f_{\text{P.D.F.}} \otimes S(\Lambda^2_{\text{QCD}}) \otimes J(Q^2) \otimes H_{\text{QCD}}(Q^2)$

- A.V.Manohar, Phys.Rev.D68.11401912003

Operator

$$J^\mu_{\text{QCD}}(x) = \bar\psi(x)\gamma^\mu\psi(x)$$



$$J^\mu_{\text{eff}} = \bar\chi_{\bar n, \bar\omega'} \gamma^\mu_\perp \chi_{n,\omega}(x)$$

$$J^\mu_{\text{QCD}} \simeq \sum_{\omega,\bar\omega'} e^{\frac{i}{2}\bar\omega' x^-} e^{\frac{i}{2}\omega x^+} c(\omega, \bar\omega', \mu_f, \mu) J^\mu_{\text{eff}}(x)$$

DIS Factorization: Matching QCD onto SCET$_{\text{I}}$

- Cross Section $\omega_{\mu\nu} = \frac{1}{\pi}\text{Im}T_{\mu\nu}(p,q)$ $T_{\mu\nu}(p,q) = \frac{1}{2}\sum_{\text{spin}}\left\langle p\middle|\hat{T}_{\mu\nu}(q)\middle|p\right\rangle$

$$\hat{T}^{\text{QCD}}_{\mu\nu} = i\int d^4x e^{iq\cdot x}T[J_\mu(x)J_\nu(x)]$$

$\downarrow$ matching

$$\hat{T}^{\text{eff}}_{\mu\nu} = i\sum_{\omega,\omega'}\int d^4x e^{iq\cdot x}e^{-\frac{i}{2}\bar{\omega}\cdot x^-}e^{-\frac{i}{2}\omega'\cdot x^+}c^*(\omega',\bar{\omega})c(\omega,\bar{\omega}')$$
$$\times T[\bar{\chi}_{n,\omega'}\gamma^\perp_\mu\chi_{\bar{n},\bar{\omega}}(x)\bar{\chi}_{\bar{n},\bar{\omega}'}\gamma^\perp_\mu\chi_{n,\omega}(0)]$$

$\downarrow$ Decouple usoft modes

$$= -\frac{i}{4}g^\perp_{\mu\nu}\sum_\sigma\sum_{\substack{\omega,\omega'\\ \bar{\omega},\bar{\omega}'}}\delta_{\bar{\omega},-Q}\delta_{\omega',Q}c^*(Q,-Q)c(\omega,\bar{\omega})\int d^4x\frac{1}{N_c}$$
$$\times\boxed{\left\langle 0\middle|T[\bar{\chi}_{\bar{n},\omega}(0)\frac{\not{n}}{2}\chi_{\bar{n},\bar{\omega}}(x)]\middle|0\right\rangle} \rightarrow \text{Jet function}$$
$$\times\boxed{\left\langle \rho,\sigma\middle|T[\bar{\chi}_{n,\omega}(x)\frac{\not{\bar{n}}}{2}\chi_{n,\omega}(0)]\middle|\rho,\sigma\right\rangle}\times\boxed{\frac{1}{N_c}\left\langle 0\middle|T[Y^+_n(x)\hat{Y}_{\bar{n}}(x)\hat{Y}^+_{\bar{n}}(0)Y_n(0)]\middle|0\right\rangle}$$

$\downarrow$ Traditional P.D.F.

$\swarrow$ soft function

Decoupling Soft & Collinear Modes in SCET

Collinear Field Redefinition: quark $\xi_{n,p}(x) = Y(x)\xi_{n,p}^{(0)}(x)$; gluon $A_{n,p} = YA_{n,p}^{(0)}Y^{+}$

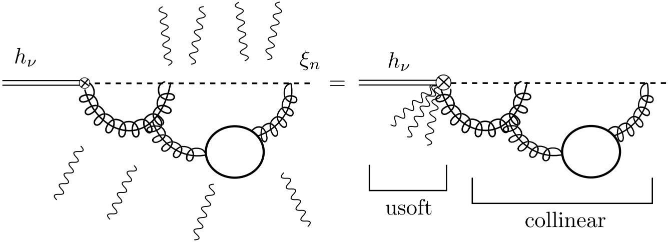
usoft $Y(x) = P\exp\left(ig\int_{-\infty}^{0} dsn\cdot A_{\mu s}^{a}(x+ns)T^{a}\right) \Rightarrow$ Collinear gluon $W = YW^{(0)}Y^{+}$

$$\mathcal{L}_{\xi\xi}^{(0)} = \bar{\xi}_{n,p'}\frac{\bar{n}}{2}[in\not{D}+\dots]\xi_{n,p} = \bar{\xi}_{n,p'}^{(0)}\frac{\bar{n}}{2}\left[Y^{+}in\cdot D_{\mu s}Y + Y^{+}(Yg\bar{n}\cdot A_{n}Y^{+})Y+\dots\right]\xi_{n,p}^{(0)}$$
$$= \bar{\xi}_{n,p'}^{(0)}\frac{\bar{n}}{2}[in\cdot\partial + g\bar{n}\cdot A_{n}+\dots]\xi_{n,p}^{(0)}$$

All $n\cdot A_{us}$'s disappear!

$$J = \bar{\xi}\omega\Gamma h_{\nu} = \bar{\xi}^{(0)}Y^{+}YW^{(0)}\Gamma h_{\nu} = (\bar{\xi}^{(0)}W^{(0)})\Gamma(Y^{+}h_{\nu})$$

$$J = (\bar{\xi}_{n}\omega)\Gamma(\omega^{+}\xi_{n}) = \bar{\xi}^{(0)}\omega^{(0)}Y^{+}Y\Gamma(W^{+(0)}\xi_{n}^{(0)})$$



DIS Factorization: Matching $\mathrm{SCET_I}$ onto $\mathrm{SCET_{II}}$

usoft $\sim \Lambda^2_{\mathrm{QCD}}$

- DIS hadronic tensor in $\mathrm{SCET_I}$

$$W^{\mu\nu}_{\mathrm{SCET_I}} = -g^{\mu\nu}_{\perp} H(Q;\mu_f,\mu) \int dr\, dl\, \boxed{J_{\bar n}(r,\mu)}\,\boxed{S(l,\mu)}$$

$$\boxed{\begin{array}{l} \times \int \frac{dn\cdot x}{4\pi} e^{-\frac{i}{2}(r+l)n\cdot x} \\ \times \frac{1}{2}\sum_\sigma \delta_{\bar n,\tilde p,Q} \left\langle h_n(\rho,\sigma) \middle| \bar\chi_n(n\cdot x) \frac{\bar{\not n}}{2} \delta_{\bar{\mathcal{P}},2Q} \chi_n(0) \middle| h_n(\rho,\sigma) \right\rangle \end{array}}$$

Initial State Collinear Function $C_n \sim \Lambda_{\mathrm{QCD}}$

Final State Jets : $Q^2\left(\frac{1}{x}-1\right) \sim Q\Lambda_{\mathrm{QCD}} \rightarrow$ offshellness in $\mathrm{SCET_{II}}$
becoming a coefficient at scale $\mu_c \sim \sqrt{Q\Lambda_{\mathrm{QCD}}}$ in $\mathrm{SCET_{II}}$

DIS Scattering Factorization Matching SCETI to SCETII

- DIS hadronic Tensor in $\mathrm{SCET_{II}}$

$$W^{\mu\nu}_{\mathrm{SCET_{II}}} = -g_\perp^{\mu\nu} H(Q;\mu_f;\mu_c)\int dl\, J_{\bar n}(l;\mu_c,\mu)\,\phi_q^{n_s}\left(Q\left(\frac{1}{x}-1\right)+l;\mu\right)$$

$$\phi_q^{\mathrm{ns}} = S(l;\mu,\nu)\,\delta_{\bar n,\tilde p,Q}\,\mathcal{Z}_n(\mu,\nu)$$

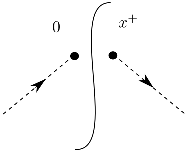
usoft mode becomes
soft mode $p_s^2 \sim \Lambda^2_{\mathrm{QCD}}$

initial usoft mode
become collinear modes
$p_c^2 \sim \Lambda^2_{\mathrm{QCD}}$

one more scale ν to separate soft & collinear modes

$$\delta_{\bar n,\tilde p,Q}\,\mathcal{Z}_n(\mu,\nu) = \int \frac{dn\cdot x}{4\pi}\, e^{-\frac{i}{2}(r+l)n\cdot x} \times \frac{1}{2}\sum_\sigma \delta_{\bar n,\tilde p,Q}\left\langle h_n(\rho,\sigma)\middle|\bar\chi_n(n\cdot x)\frac{\not{\bar n}}{2}\delta_{\bar p,2Q}\,\chi_n(0)\middle|h_n(\rho,\sigma)\right\rangle$$

Collinear Function: Feynman Rules & Tree Level Result



n collinear quark
Incoming proton
Light-cone direction n^μ
Large light-cone momentum
$\tilde{P}_n = \bar{n} \cdot P = Q$ since
$x \to 1 \quad x = \frac{Q^2}{2P\cdot q}$

$$C_n(P_n, Q, k^-) = \int \frac{dx^+}{4\pi} e^{-\frac{i}{2}k^- x^+} \times \frac{1}{2} \sum_\sigma \left\langle h_n(\rho,\sigma) \middle| \bar{\chi}_n(x^+) \frac{\bar{\not{n}}}{2} \delta_{\bar{P},2Q} \chi_n(0) \middle| h_n(\rho,\sigma) \right\rangle$$

$$\text{at } \mathcal{O}(\alpha_s^0) \;\; = \int \frac{dx^+}{4\pi} e^{-\frac{i}{2}k^- x^+} \frac{1}{2} \sum_\sigma \left\langle h_n \middle| \bar{\xi}_{n,P_1}(x^+) \frac{\bar{\not{n}}}{2} \delta_{\bar{P},2Q} \chi_{n,P_2}(0) \middle| h_n \right\rangle$$

$$= \int \frac{dx^+}{4\pi} e^{-\frac{i}{2}k^- x^+} e^{\frac{i}{2}\bar{n}\cdot P_n x^+} \delta_{\tilde{P}_n \cdot \bar{n},Q} \frac{1}{2} \sum_\sigma \bar{\xi}_n^\sigma \frac{\bar{\not{n}}}{2} \xi_n^\sigma$$

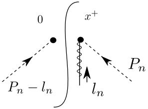
$$= \delta_{\bar{n},\tilde{P}_n,Q} \delta(\bar{n} \cdot P_r - k) m_0 = \delta_{\bar{n}\cdot\tilde{P}_n,Q} \delta(k) m_0$$

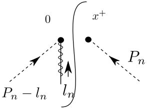
$\bar{\chi}_{n,P}, \chi_{n,P}$ are only defined with residue momentum

$$\bar{\chi}_{n,P}(n \cdot x) = e^{i(i\bar{n}\cdot\partial)x^+/2} \bar{\chi}_n(0) e^{-i(i\bar{n}\cdot\partial)x^+/2}$$

$$\xi_{n,P} | P_n, \sigma\rangle = \delta_{\tilde{P}_n \cdot \bar{n},Q}$$

Collinear Function: No Real Emission at $\mathcal{O}(\alpha_s)$

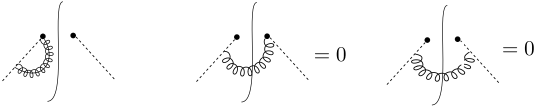




$$C_n(P_n, Q+k^-) = \int \frac{dx^-}{4\pi} \frac{1}{2} \sum_\sigma \langle P_n, \sigma \big| \bar{\xi}_{n,P_1} e^{\frac{i}{2}x^+ P_r^-} e^{-\frac{i}{2}x^+ l_n^{r-}} (ig) \int_{-\infty}^0 ds \bar{n}\cdot A_n(s_n^-)$$
$$\frac{\bar{\not{n}}}{2} \delta_{\bar{P},2Q} \xi_{n,P_2} + \bar{\xi}_{n,P_1} e^{\frac{i}{2}x^+ P_r^-} \frac{\bar{\not{n}}}{2} \delta_{\bar{P},2Q} (-ig) \int_{-\infty}^0 ds \bar{n}\cdot A_n(s\bar{n}) \xi_{n,P_2} \big| P_n - l_n, \sigma; l_n; r\rangle$$
$$= \frac{igT^A \epsilon(\lambda)\cdot\bar{n}}{\bar{n}\cdot l - i\epsilon} [\delta(P_r^- - l_n^{-r} - k^-) \delta_{\tilde{P}_n\cdot\bar{n} - \tilde{l}_n\cdot\bar{n},Q} - \delta(P_r^- - k^-) \delta_{\tilde{P}_n\cdot\bar{n},Q}] \cdot \delta_{\tilde{P}_n\cdot\bar{n},Q} \cdot m_0$$

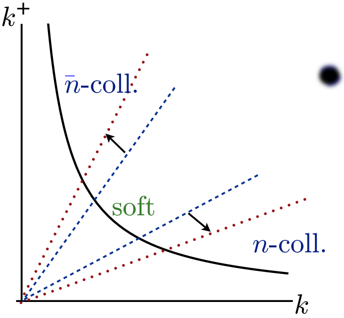
$\delta_{\tilde{P}_n\cdot\bar{n},Q} \delta_{\tilde{P}\cdot\bar{n} - \tilde{l}_n\cdot\bar{n},Q} \to \delta_{\tilde{l}_n\cdot\bar{n},0}$ collinear gluon becomes soft gluon

At Endpoint Region, No real collinear emission is allowed!



$$im_{(a)} = im_0 (2g_s^2(t) \delta_{n\cdot\tilde{P},Q} \delta(k) \mu^{2\epsilon}$$
$$\sum_{\bar{n}\cdot\tilde{l}\neq 0} \int \frac{d^D l}{(2\pi)^D} \frac{1}{\bar{n}\cdot q} \frac{\bar{n}\cdot(p-q)}{(p-q)^2 + i\epsilon} \frac{1}{q^2 + i\epsilon}$$

Rapidity Divergences



Chin, Jain, Neil, Rothstein arXiv:1202.0814

- $I_s = \int d^dk \frac{1}{(k^2-M^2)} \frac{1}{(-n\cdot k+i\epsilon)} \frac{1}{-\bar{n}\cdot k+i\epsilon)}$

  integrating over $k_\perp$, IR finite orver M^2

  $$\sim \int [d^2k](n\cdot k\bar{n}\cdot k - M^2)^{-2\epsilon} \frac{1}{(-n\cdot k+i\epsilon)} \frac{1}{(-\bar{n}\cdot k+i\epsilon)}$$

  along hyperbola $n\cdot k\bar{n}\cdot k \sim M^2$, I_s diverges

  $n\cdot k/\bar{n}\cdot k \to \infty$ or $n\cdot k/\bar{n}\cdot k \to 0$

  Because soft & collinear modes are mixed

Rapidity Regulator v.s. Delta Regulator

Collinear Wilson Lines

$$W_n = \sum_{\text{perms}} \exp\left[-\frac{g}{\bar{n}\cdot p}\frac{|\bar{n}\cdot p|^{-\eta}}{\nu^{-\eta}}\bar{n}\cdot A_n\right] \quad W_n = \sum_{\text{perms}} \exp\left[-\frac{g}{\bar{n}\cdot p+\Delta_k}\bar{n}\cdot A_n\right]$$

Soft Wilson Lines

$$S_n = \sum_{\text{perms}} \exp\left[-\frac{g}{n\cdot p}\frac{|\bar{n}\cdot p-n\cdot p|^{-\eta/2}}{\nu^{-\eta/2}}n\cdot A_s\right] \quad S_n = \sum_{\text{perms}} \exp\left[-\frac{g}{n\cdot p+\delta_l}n\cdot A_s\right]$$

- Gauge Invariance?
- Clearly delineates sectors?
- Preserving Factorization Theorem
  - A universal definition for generalized soft and jet function?

Collinear Function and Rapidity Regulator

Nonzero virtual emission

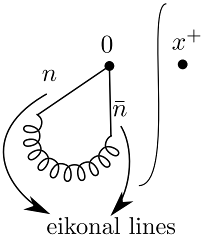
$$
\begin{aligned}
C_n = 2m_a &= m_0\delta_{\bar{n}\cdot\tilde{p},Q}\delta(k)(4g_s^2c_F)\mu^{2\epsilon}\nu^{\eta}\int\frac{d^Dq}{(2\pi)^D}\frac{1}{\bar{n}\cdot q}\frac{\bar{n}\cdot(p+q)|\bar{n}\cdot q|^{-\eta}}{((p+q)^2+i\epsilon)(q^2-\boxed{m_g^2}+i\epsilon)}\\
&= m_0\delta_{\bar{n}\cdot\tilde{p},Q}\delta(k)\frac{\alpha_sc_F}{\pi}\left\{\frac{e^{\epsilon\gamma_E}\Gamma(\epsilon)}{\eta}\left(\frac{\mu^2}{m_g^2}\right)^{\epsilon}+\frac{1}{\epsilon}\left[1+\ln\frac{\nu}{\bar{n}\cdot p}\right]\right.\\
&\quad\left.+\ln\frac{\mu^2}{m_g^2}\ln\frac{\nu}{\bar{n}\cdot p}+\ln\frac{\mu^2}{m_g^2}+1-\frac{\pi^2}{6}\right\}
\end{aligned}
$$

zero-bin?

$$
\begin{aligned}
C_n &\xrightarrow{\text{soft limit}} C_{n\phi} = 2m_{a\phi}\\
&= m_0\delta(k)\delta_{\bar{n}\cdot\tilde{P},Q}(2g_s^2c_F)\mu^{2\epsilon}\int d^dq\frac{1}{(\bar{n}\cdot q)^{1+\eta}}\frac{1}{q^++i\epsilon}\frac{1}{q^2-m_g^2+i\epsilon}\\
&\sim \int_0^{\infty}dq^-\frac{1}{(q^-)^{1+\eta}}\ \text{scales}\ \to 0
\end{aligned}
$$

Rapidity Regulator: automatically cutting the soft bin off

Rapidity Regulator and Virtual Soft Function



$$S_\nu = -i4g_s^2 c_F \delta(l)\mu^{2\epsilon}\nu^\eta \int d^dk \frac{|2k^{(3)}|^{-\eta}}{(k^2-\boxed{m_g^2}+i\epsilon)(k^-+i\epsilon)(k^++i\epsilon)}$$

$$= \delta(l)\frac{2\alpha_s}{\pi}\left[-\frac{e^{\epsilon\gamma_E}\Gamma(\epsilon)}{\eta}\left(\frac{\mu}{m_g}\right)^{2\epsilon} + \frac{1}{2\epsilon^2} + \frac{1}{\epsilon}\ln\frac{\mu}{2} + \ln^2\frac{\mu}{m_g} - \ln\frac{\mu^2}{m_g^2}\ln\frac{\nu}{m_g} + \frac{\pi^2}{24}\right]$$

zero-bin: taking n, $\bar{n}$ collinear part from this soft function

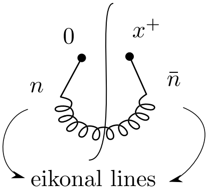
$\bar{n}$-collinear limit: $S^{\bar{n}}_{\nu\phi} = -(4ig^2c_F)\mu^{2\epsilon}\nu^\eta \int d^dk|k^-|^\eta$

$$\frac{1}{(k^2-m_g^2+i\epsilon)(k^++i\epsilon)(k^-+i\epsilon)}$$

$$\sim \int_{-\infty}^{0}\frac{dk^-}{2\pi}|k^-|^\eta|k^-|^{-1} \xrightarrow{\text{dim-reg}} 0$$

$$S^n_{\nu\phi} = S^{\bar{n}}_{\nu\phi}$$

Rapidity Regulator and Real Soft Function



- $S_R = -4c_F g_s^2 \mu^{2\epsilon} \nu^\eta \int \frac{d^D k}{(2\pi)^{D-1}} \delta(k^2 - m_g^2)\delta(l - k^+) \frac{|k^+ - k^-|^{-\eta}}{(k^+ + i\epsilon)(k^- + i\epsilon)}$
- $k^+ \gg k^-$ n-collinear limit,

$$S_{n\phi} = -4c_F g_s^2 \mu^{2\epsilon} \nu^\eta \int \frac{d^D k}{(2\pi)^{D-1}} \delta(k^2 - m_g^2)\delta(l - k^+) \frac{|k^+|^{\eta}}{(k^+ + i\epsilon)(k^- + i\epsilon)}$$

$k^- \gg k^+$ $\bar{n}$-collinear limit,

$$S_{\bar{n}\phi} = -4c_F g_s^2 \mu^{2\epsilon} \nu^\eta \int \frac{d^D k}{(2\pi)^{D-1}} \delta(k^2 - m_g^2)\delta(l - k^+) \frac{|k^-|^{\eta}}{(k^+ + i\epsilon)(k^- + i\epsilon)}$$

- Because of the measurement $\delta(l - k^+)$ k^+ is fixed

(1) $S_n \neq S_{\bar{n}} \neq 0$ as zero as virtual soft function

(2) Expanding S_R, the difference between S_n & S_R

is at $\mathcal{O}\left(\frac{m_g^2}{l^2}\right)$; $S_R = S_n$ at $\mathcal{O}\left(\left(\frac{m_g^2}{l^2}\right)^0\right)$

$$\therefore\ S_R - S_n - S_{\bar{n}} = -S_{\bar{n}} = 2\frac{\alpha_s}{\pi}\omega^2 \left\{ \left[\frac{1}{2} \frac{e^{\epsilon\gamma_E}\Gamma(\epsilon)}{\eta} \left(\frac{\mu}{m_g}\right)^{2\epsilon} - \frac{1}{2\epsilon^2} + \frac{1}{2\epsilon} \ln \frac{\nu}{\mu^2} \right.\right.$$
$$\left.\left. - \ln^2 \frac{\mu}{m_g} + \ln \frac{\mu}{m_g} \ln \frac{\nu}{m_g^2} + \frac{\pi^2}{24} \right] \delta(l) + \left[\frac{1}{2\epsilon} + \ln \frac{\mu}{m_g} \right] \frac{1}{l_+} \right\}$$

Renormalization Consistency Condition

- Renormalized Collinear and Soft Function

$$C_n(Q+k)^R = Z_n^{-1} C_n(Q+k)^B \quad S(l)^R = \int dl' Z_s(l-l')^{-1} S(l')^B$$

$\Rightarrow$ One loop collinear and soft counter term

$$Z_n = 1 + \frac{\alpha_s c_F}{\pi}\left[\frac{e^{\epsilon\gamma_E}\Gamma(\epsilon)}{\eta}\left(\frac{\mu}{m_g}\right)^{2\epsilon} + \frac{1}{\epsilon}\left(\frac{3}{4} + \ln\frac{\nu}{\bar{n}\cdot p}\right)\right]$$

$$Z_s = \delta(l) + \frac{\alpha_s c_F}{\pi}\left\{-\frac{e^{\epsilon\gamma}\Gamma(\epsilon)}{\eta}\left(\frac{\mu}{m_g}\right)^{2\epsilon}\delta(l) + \frac{1}{\epsilon}\left[\frac{1}{(l)_+} - \ln\frac{\nu}{\mu}\delta(l)\right]\right\}$$

- Consistency Condition for Counter Terms, From the Factorization in SCET

Becher, Neubert, Pecjak, JHEP 0701, 076 (2007)

Non-trivial check! $\quad Z_H Z_{J_{\bar{n}}}(l) = Z_n^{-1} Z_s^{-1}(l)$

Renormalization Consistency Condition

A.V.Manohar, Phys.Rev. D68, 114019(2003)

$$Z_{J_{\bar{n}}}(l) = \delta(l) + \frac{\alpha_s c_F}{4\pi}\left[\left(\frac{4}{\epsilon^2} + \frac{3}{\epsilon} - \ln\frac{n\cdot P_Q}{\mu}\right)\delta(l) - \frac{4}{\epsilon}\frac{1}{l^+}\right]$$

C.W.Bauer, C.Lee, A.V.Manohar, M.B.Wise, Phys.Rev.D70, 034014 (2004)

$$Z_H(l) = 1 - \frac{\alpha_s c_F}{2\pi}\left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{2}{\epsilon}\ln\frac{\mu^2}{Q^2}\right)$$

$$Z_c^{-1} Z_s^{-1}(l) = \delta(l) - \frac{\alpha_s}{4\pi}\left\{\frac{1}{\epsilon} + \frac{1}{l_+} + \frac{3}{4}\delta(l) - \frac{1}{\epsilon}\ln\frac{n\cdot P_n}{\mu}\right\}$$

At one loop: $Z_{J_{\bar{n}}} Z_H = \delta(l) + \frac{\alpha_s c_F}{\pi}\left\{\left[-\frac{3}{4\epsilon} + \frac{1}{3}\ln(\frac{\bar{n}\cdot p}{\mu})\right]\delta(l) - \frac{1}{\epsilon}\frac{1}{l_+}\right\}$

Agree with $Z_n^{-1} Z_s^{-1}(l)$

Double Running in Infrared Scale & Rapidity Scale

- μ anomalous (Infrared Scale)

$$\gamma_n^\mu(\mu,\nu) = \frac{2\alpha_s c_F}{\pi}\left(\frac{3}{4} + \ln\frac{\nu}{\bar{n}\cdot P_n}\right);\ \gamma_s^\mu(l;\mu,\nu) = \frac{2\alpha_s c_F}{\pi}\left[\frac{1}{l_+} - \ln\frac{\nu}{\mu}\delta(l)\right]$$

- ν anomalous (Rapidity Scale)

$$\gamma_n^\nu(\mu,\nu) = \frac{\alpha_s c_F}{\pi}\ln\frac{\mu^2}{m_g^2};\ \gamma_s^\nu(\mu,\nu) = -\frac{\alpha_s c_F}{\pi}\ln\frac{\mu^2}{m_g^2}$$

Because of the rapidity scale ν, in γ^μ, the 'usual' anomalous, we can

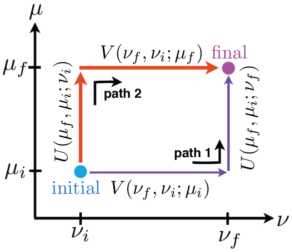
(1) using ν to signaling the endpoint region

$$\nu_c \sim Q \text{ in } \gamma_n^\mu \quad \nu_s \sim Q\left(\frac{1}{x} - 1\right) \text{ in } \gamma_s^\mu$$

(2) $\gamma^\mu = \gamma_n^\mu + \gamma_s^\mu = \frac{2\alpha_s c_F}{\pi}\left[\frac{3}{4} + \frac{1}{l_+} + \ln\frac{\mu}{\bar{n}\cdot P_n}\right] \qquad \gamma^\nu = 0$

ν cancels as expected! $\quad$ ν expands where $\ln\frac{\mu}{\bar{n}\cdot P_n} \sim \ln\frac{\mu}{Q}$ comes from

Double Running in Infrared Scale & Rapidity Scale



- One-loop μ-running factor for collinear & soft function

$$C_n(Q_K;\mu,\nu_c) = \mathcal{U}(\mu,\mu_0,\nu_c)C_n(Q-K;\mu_0,\nu_c)$$

$$\mathcal{U}(\mu,\mu_0,\nu_c) = e^{\frac{3}{4}\omega(\mu,\mu_0)}\left[\frac{\nu_c}{\bar{n}\cdot p}\right]^{\omega(\mu,\mu_0)}$$

$$S(l;\mu,\nu_s) = \int dr\,\mathcal{U}(l-r;\mu,\mu_0,\nu_s)S(l;\mu_0,mu_s)$$

$$\mathcal{U}(l-r;\mu,\mu_0,\nu_s) = \frac{(e^{2\gamma_E}\nu_s)^{-\omega(\mu,\mu_0)}}{\Gamma(\omega(\mu,\mu_0))[(l-r)^{1-\omega(\mu,\mu_0)}]_+}$$

$$\omega(\mu,\mu_0) = \frac{4c_F}{\beta_0}\ln\left[\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right]$$

- However, running in ν is not perturbative since $\gamma_n^\nu, \gamma_s^\nu$ depend on m_g^2

$$S(l;\mu_s,\nu) = V(\mu_s,\nu,\nu_0)S(l;\mu_s,\nu_0)$$

$$V(\mu_s,\nu,\nu_0) = \left[\frac{\nu}{\nu_0}\right]^{\omega(\mu_s,m_g)}$$

Convolving with non-perturbative P.D.F. part which can absorb m_g^2

Collinear Function with Delta Regulator

Virtual Contribution

$$C_n^\nu = 2m_a = m_0 \delta_{\bar{n}\cdot\tilde{p},Q} \delta(k) (4g_s^2 c_F) \mu^{2\epsilon} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^- - \delta_1 + i\epsilon} \frac{1}{q^+ - \frac{q_+^2 + \Delta_2 - i\epsilon}{p^- + q^-}} \frac{1}{q^+ - \frac{q_\perp^2 + m_g^2 - i\epsilon}{q^-}} \frac{1}{q^-}$$

$$= \left(\frac{\alpha_s c_F}{\pi}\right) \delta(k^-) \Big\{ -\frac{1}{\epsilon} \left[\log \frac{\delta_1}{p^-} + 1\right] - \log \frac{\mu^2}{m_g^2} \left(\log \frac{\delta_1}{p^-} + 1\right)$$

$$- \left[\ln\left(1 - \frac{\Delta_2}{m_g^2}\right) \ln \frac{\Delta_2}{m_g^2} + 1 - \frac{\Delta_2/m_g^2}{\Delta_2/m_g^2 - 1} \ln \frac{\Delta_2}{m_g^2} + \mathrm{Li}_2\left(\frac{\Delta_2}{m_g^2}\right) - \frac{\pi^2}{6}\right] \Big\}$$

zero-bin of virtual contribution, never being zero again $\frac{\Delta_2}{p^-} = \delta_2$

$$C_{n\phi}^\nu = m_0 \delta_{\bar{n}\cdot\tilde{p},Q} \delta(k) (4g_s^2 c_F) \mu^{2\epsilon} \int d^d q \frac{1}{q^- - \delta_i + i\epsilon} \frac{1}{q^+ - \frac{\Delta_2}{p^-} + i\epsilon} \frac{1}{q^+ - \frac{q_1^2 + m_g^2 - i\epsilon}{q^-}} \frac{1}{q^-}$$

$$= \frac{\alpha_s c_F}{\pi} \delta(l) \left\{ \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \log \frac{\mu^2}{\delta_1 \delta_2} + \log \frac{\mu^2}{m_g^2} \log \frac{\delta_1 \delta_2}{m_g^2} + \frac{1}{2} \log^2 \frac{\mu^2}{m_g^2} \right.$$

$$\left. + \frac{1}{2} \log^2 \left(\frac{\delta_1 \delta_2}{m_g^2} - 1\right) + \mathrm{Li}_2 \left(\frac{1}{1 - \frac{\delta_1 \delta_2}{m_g^2}}\right) + \frac{7\pi^2}{12} \right\}$$

Real Contribution, only soft momentum allows to traverse the cut

$$C_n^R = 2m_b + 2m_c = C_{n\phi}^R = 2m_b + 2m_c = 0$$

Soft Function with Delta Regulator

Virtual Contribution

$$S^{\nu} = \frac{C^{\nu}_{n\phi}}{m_0 \delta_{\bar{n}\cdot\tilde{p},Q}} = \delta(l)(4g_s^2 c_F)\mu^{2\epsilon}\int d^d q \frac{1}{q^- - \delta_1 + i\epsilon}\frac{1}{q^+ - \boxed{\delta_2} + i\epsilon}\frac{1}{q^2 - m_g^2 + i\epsilon}$$

$$\delta_2 = \frac{\Delta_2}{p^-}$$

Collinear-bin subtraction, $\bar{n}$-direction $k^- \gg k^+$, Rapidity Divergence rises

$$S^{\nu}_{\bar{n}\phi} = (2ig_s^2 c_F)\delta(l)\mu^{2\epsilon}\int d^d k \frac{1}{k^2 - m_g + i\epsilon}\frac{1}{k^+ - \boxed{\frac{\Delta_2 - i\epsilon}{k^- + \delta_3}}}\frac{1}{k^- - \delta_1 + i\epsilon}$$

δ_3 ∼collinear scale

Regulating rapidity divergence,

Destroying factorization!

Real Contribution

$$S^R = (4\pi g^2 c_F)\mu^{2\epsilon}\int d^{4-2\epsilon}k_\perp \delta(k^2 - m_g^2)Q(k^0)\delta(l - k^+)\frac{1}{k^+ - \delta_2}\frac{1}{k^- - \delta_1}$$

because of the measurement function, not containing an obvious rapidity divergence

However in $S^R_{\bar{n}\phi}, S^R_{n\phi}$ $\quad \delta_2 \to \frac{\Delta_2 - i\epsilon}{k^{\pm} + \delta_3}$

Renormalization with Delta Regulator at Endpoint Region

Only Divergent Part

$$\frac{\alpha_s c_F}{\pi}\left\{\frac{1}{\epsilon^2}\delta(l^+)+\frac{1}{\epsilon}\ln\left(\frac{\mu}{\delta_1}\right)\delta(l^+)-\frac{1}{\epsilon}\frac{1}{[l^+]_+}+\frac{2}{\epsilon}\ln\left(\frac{\delta_1}{\delta_3}\right)-\frac{2}{\epsilon}\ln\left(\frac{\delta_1}{\delta_3}\right)\right.$$

soft

$$\left.-\frac{1}{\epsilon^2}\delta(l^+)-\frac{1}{\epsilon}\ln\left(\frac{\mu^2}{\delta_1 \cdot p^-}\right)\delta(l^+)+\frac{1}{\epsilon}\delta(l^+)+\mathcal{O}(1)\right\}\quad\text{collinear}$$

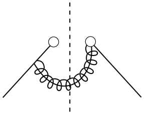
$$=\frac{\alpha_s c_F}{\pi}\left\{-\frac{1}{\epsilon}\frac{1}{[l^+]_+}+\frac{1}{\epsilon}\ln(\frac{p^-}{\mu})\delta(l^+)+\frac{1}{\epsilon}\delta(l^+)\right\}$$

with $z_\xi=\frac{\alpha_s c_F}{4\pi}\frac{1}{\epsilon}$ wave function renormalization

$$\gamma=2\left(\frac{\alpha_s c_F}{\pi}\right)\left\{\frac{1}{[l^+]_+}-\ln(\frac{p^-}{\mu})\delta(l^+)-\frac{3}{2}\delta(l^+)\right\}$$

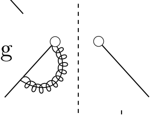
A suggestion to a New Definition of PDF?

- *Collinear Function*



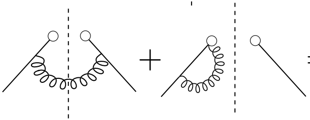
$= 0$

only containing



$\neq 0$ initial states information

- Soft function



$\neq 0$

containing both initial states & final states information

- $W_{\rm eff} \sim C^2(Q,Q,\mu_f,\mu) \otimes f^{\rm P.D.F} \otimes J_{\rm eff} \otimes S_{\rm eff}$

$\downarrow$

$\boxed{\text{should be only sensitive to initial states}}$

$$f^n_{\rm P.D.F.}(z;\mu) \equiv \frac{1}{2}\sum_\sigma \left\langle h_n(\rho,\sigma)\middle|\bar\chi_n(0)\frac{\bar{n}\!\!\!/}{2}\chi_n(0)\middle|h_n(\rho,\sigma)\right\rangle \times \int \frac{dn\cdot x}{4\pi}e^{\frac{i}{2}Qzn\cdot x}\frac{1}{N_c}\left\langle 0\middle|\mathrm{Tr}\left(\bar{T}[Y_n^+ Y_{\bar n}(n\cdot x)]T[Y_{\bar n}^+ Y_n(0)]\right)\middle|0\right\rangle$$

containing both collinear and soft factor, satsifying DGLAP type running

Summary & Future Project

- DIS End point Factorization QCD → $\mathrm{SCET_I} \to \mathrm{SCET_{II}}$
- Rapidity Divergence: more specific in future

  Rapidity regulator v.s. Delta regulator

  ↙

  Gauge Invariance; Clearly Separation of Soft & Collinear;

  Universally defining in Soft and Collinear Wilson Lines (Keeping Factorization)
- Double Running in Infrared & Rapidity Scales

  More Information from Endpoint region
- New P.D.F.? Phenomenology Analysis, in future